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# Vibrations of viscoelastic, power-law solids[1]

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## ABSTRACT

*A fractal tree of nested viscoelastic boxes is proposed to describe the elastic after-effect in some solids that show evidence of a hierarchical structure, such as clay, some rocks, concrete and polycrystalline alloys. A functional equation is derived and solved for the Laplace's transform of the creep kernel of the whole hierarchy. Inverting the transform, Hopkinson's power law is obtained as a (fairly) accurate approximation, after an initial time interval. The exponent in the power law is obtained as a well-defined function of the branching numbers and scaling parameters of the viscoelastic hierarchy. Then we consider a composite solid with both elastic (impact) strain response and power law type elastic after-effect, for an applied stress step. The stress response to an applied strain is given in terms of Rabotnov's fractional exponential functions. The stretched exponential stress relaxation to an applied strain step is derived as a fairly close approximation. According to the present mathematical model, for a given composite solid, the stretch parameter of the stretched exponential should be equal to the exponent of the power law. The experimental lower bound (0) and upper bound (1) for this parameter are derived from the theory. A viscoelastic generalization an equation proposed by Love to describe the longitudinal vibrations of bars of uniform cross-section and including the effects of lateral inertia, is briefly studied to describe mechanical vibrations in power-law solid bars.*

***Key Words*: Mechanical vibrations of solids, elastic after-effect, fractal hierarchy, power law strain, stretched exponential stress relaxation, functional equations, Rabotnov's fractional exponential function, Love's equation for viscoelastic beams, non-destructive testing and diagnosis, resonance experiments**.

## INTRODUCTION

In the framework of continuum mechanics, a (fairly) detailed study of mechanical vibrations can be done, using an analytical approach, if the body is homogeneous, isotropic and with linear viscoelasticity (linear hereditary properties), and if a simplified enough mathematical model is used.

When the method of separation of variables can be applied to solve the dynamic equations of the mathematical model, displacement fields can be expressed as a series in terms of space Eigen functions weighted by the corresponding time functions. With this construction both, boundary conditions and initial conditions can be satisfied.

It is an interesting and general result that the time functions $T(t)$ are the solutions of linear integral-differential equations like this one (Rabotnov, 1980):

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[1] **This archive is a modern version of the paper. The original article was written with an IBM electric machine and the formulae were handwritten. The presentation was done using a blackboard in the old way. Transparent sheets ("filminas") for text projection and formulas were available, but of course there was no power point in those times.**

$$\frac{d^2}{dt^2}T(t)+\omega^2\left(T(t)-\int_{-\infty}^{t}L(t-t')T(t')dt'\right)=\omega^2 q(t) \tag{1}$$

The frequency $\omega$ appears during the process of solving a suitable boundary value problem for the space functions.

The function $q(t)$ stems from the projection of the density of the distributed external forces applied to the body onto an Eigen function and is zero if these forces are absent.

The kernel $L(t)$ characterizes the hereditary (delayed viscoelastic) properties of the material.

As an example, let us consider the following equation derived by Love to describe the longitudinal vibrations of bars of uniform cross-section and including the effects of lateral inertia, to describe mechanical vibrations in power-law solid bars (Love, 1927):

$$\rho\,\frac{\partial^2 s(t,x)}{\partial t^2}=E_0\,\frac{\partial^2 s(t,x)}{\partial x^2}+\rho\left(\nu_0\, r_0\right)^2\frac{\partial^4 s(t,x)}{\partial x^2\,\partial^2 t}+\rho\, X(t,x) \tag{2}$$

The bar material is assumed to be linear elastic, homogeneous and isotropic, being $s(t,x)$ the longitudinal displacement (a function of time $t$ and spatial position $x$ along the bar axis), $\rho$ the density, $E_0$ the Young modulus, $\nu_0$ the Poisson modulus, and $X(t,x)$ gives the distribution of longitudinal external forces.

The characteristic radius $r_0$ is given by a double integral taken over the bar cross-section $S_{cs}$, being $A(S_{cs})$ its area:

$$r_0^{\,2}=\frac{1}{A(S_{cs})}\iint_{S_{cs}}\left(y^2+z^2\right)dy\,dz \tag{3}$$

The origin $y=0$ $z=0$ is located on the bar axis.

The viscoelastic generalization of equation (2) can be done as follows.

Let us consider a linear hereditary relation between the longitudinal scalar stress $\sigma(t,x)$ and the corresponding longitudinal scalar strain $\varepsilon(t,x)=\frac{\partial}{\partial x}s(t,x)$:

$$\sigma(t,x)=E_0\left(\varepsilon(t,x)-\int_{-\infty}^{t}L(t-t')\,\varepsilon(t',x)\,dt'\right) \tag{4 a}$$

Its inverse is:

$$\varepsilon(t,x)=\frac{1}{E_0}\left(\sigma(t,x)+\int_{-\infty}^{t}K(t-t')\,\sigma(t',x)\,dt'\right) \tag{4 b}$$

Then, a possible viscoelastic generalization of equation (2) is this one:

$$\frac{\partial^2 s(t,x)}{\partial t^2}=c_0^2\left(\frac{\partial^2 s(t,x)}{\partial x^2}-\int_{-\infty}^{t}L(t-t')\,\frac{\partial^2 s(t',x)}{\partial x^2}\,dt'\right)+\left(\nu_0\, r_0\right)^2\frac{\partial^4 s(t,x)}{\partial x^2\,\partial^2 t}+X(t,x) \tag{5}$$

In equation (5), $c_0$ is the velocity of extensional waves:

$$c_0=\sqrt{\frac{E_0}{\rho}} \tag{6}$$

Now, let us consider a particular case of the constitutive equations (4 a) and (4 b).

It is known that certain solids, that show evidence of having a hierarchical structure, like clays, some rocks, concrete and certain polycrystalline alloys, often present hyperbolic delayed strain responses (known as power law creep) to a step of stress, and exponentially stretched stress responses to a step of strain, at least very approximately, during long periods of time, after a suitable initial time interval.

After its discovery in 19th century[2], both hyperbolic and stretched exponential responses were often found studying the mechanical properties of many bodies, including glasses, polymers, rocks, concrete members, polycrystalline alloys and samples of biological tissues.

When a constant stress of magnitude $\sigma_0$ is applied at the origin of the time variable $t$, being zero before, the strain response $\varepsilon(t)$ to an applied step of stress has a linear elastic component $\varepsilon_0$ and a delayed component $\varepsilon_d(t)$: $\quad \varepsilon(t) = \varepsilon_0 + \varepsilon_d(t)$

If $J_0 = \dfrac{1}{E_0}$ is the elastic compliance modulus, the elastic response is given by $\varepsilon_0 = J_0 \cdot \sigma_0$

The delayed strain seems to follow a hyperbolic law $\varepsilon_d(t) \approx C \cdot \left(\dfrac{t}{T_\varepsilon}\right)^p \cdot \sigma_0$

Here $C$ is a positive parameter. This power law fits very accurately experimental data for times $t$ greater than the characteristic time $\tau_\varepsilon$. The parameter $p$ is always between 0 and 1.

So, the whole strain response to a step in the stress is given by:

$$\varepsilon(t) \cong \left( J_0 + C \cdot \left(\frac{t}{T_\varepsilon}\right)^p \right) \cdot \sigma_0$$

For a variable applied stress $\sigma(t)$ the delayed strain response of a linear viscoelastic power law material is given by: $\quad \varepsilon_d(t) = \int_0^t Q(t-t') \cdot \sigma(t') \cdot dt'$

In this case $Q(t) = K(t)$ is the hereditary kernel in the case of this kind of material.

If $\sigma(t) = \sigma_0$ for $t > 0$ we obtain $\varepsilon_d(t) = \sigma_0 \cdot \int_0^t Q(t') \cdot dt'$

Then $\int_0^t Q(t') \cdot dt' = C \cdot \left(\dfrac{t}{T_\varepsilon}\right)^p$ so the viscoelastic kernel is (with $\alpha = 1 - p$ and $Q(T_\varepsilon) = \dfrac{(1-\alpha) \cdot C}{T_\varepsilon}$):

$$Q(t) = Q(T_\varepsilon) \cdot \left(\frac{T_\varepsilon}{t}\right)^\alpha \qquad (7)$$

If $\sigma(t) = \sigma_0$ for $0 < t < T_0$ and zero otherwise, then the delayed strain response for $t > T_0$ is:

$$\varepsilon_d(t) = \sigma_0 \cdot \int_0^{T_0} Q(t-t') \cdot dt' = \sigma_0 \cdot C \cdot \left(\frac{t}{T_\varepsilon}\right)^{(1-\alpha)} \left( 1 - \left(1 - \frac{T_0}{t}\right)^{1-\alpha} \right)$$

But in this case, if $t \to +\infty$ the delayed strain response $\varepsilon_d(t) \to 0$

As, of course, the instantaneous elastic response disappears also, the material behaves as a solid. This behaviour justifies the usual denomination of "power law solid" applied to this kind of materials.

Now, notice that the power law response can be written $\varepsilon_d(t) = J(t) \cdot \sigma_0$ with $J(t) = J(1) \cdot t^p$. The compliance $J(t)$ verifies the following *functional equation*: $J(t_0 \cdot x) = M(t_0) \cdot F(x)$ Here $M(t_0) = t_0^p$ and $F(x) = J(1) \cdot x^p$.

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[2] **Apparently, the power law creep was discovered by W. Weber in 1836, studying the delayed strain response of the silk threads of galvanometers under a constant tensile load.**

With $t_0$ fixed, $x \in [0,1]$ and $t = t_0 \cdot x \in [0,t_0]$, *this functional equation characterizes a scale invariant response* $J(t)$ in a time interval $[0,t_0]$: $M(t_0)$ gives the amplitude and $F(x)$ gives the shape (*always the same*) of the response. Conversely, the only regular solutions of the functional equation of scale invariance are the power functions (Pipkin, 1972).

The stretched exponential stress relaxation is given by:

$$\sigma_r(t) \approx \sigma(0) \cdot \exp\left[-\left(\frac{t}{T_\sigma}\right)^p\right] \tag{8}$$

For times $t$ greater than $T_\sigma$, the stretched exponential fits to experimental data fairly accurately. The parameter $p$ also is always between 0 and 1[3].

There is another fact that is closely related with the hyperbolic response. It is the relative constancy of the quality factor corresponding to elastic wave dissipation in the Earth, for most earth materials and for wave periods comprised between $10^{-5}$ and $10^3$ seconds: it lies between 100 and 1000 (Pilant, 1979).

As will be seen here, a possible interpretation of the hyperbolic or stretched exponential behaviour of power-law solids can be obtained assuming that strain and stress responses result from the addition of partial strains or stresses that correspond to the degrees of freedom in an infinite hierarchical structure.

The purposes of the present paper are twofold:

(1)-First, to apply the elementary mechanical theory of composite materials to a suitably defined hierarchical structure of elastic elements of different sizes and shapes immersed in a viscous matrix. A given degree of freedom will be able to proceed to a relaxation process, if the precedent degree of freedom has relaxed before. As consequence, the time scale of evolution of a given level is subordinated to the time scale of the precedent level: each level responds slower than the precedent level but faster than the next one. Under the assumption that the dominant viscous dissipation occurs at the interfaces between the elastic elements and the viscous matrix, we obtain the power law creep that describes the delayed strain response of the above-mentioned composites.

In order to do that, we derive a functional equation in Laplace domain, whose solution in time domain corresponds to power law creep. This derivation follows an approach closely related with Liu's explanation of constant phase angle impedances and 1/f noises in rough electrode interfaces (Liu, 1985).

Then we show that the stretched exponential can be considered as a fairly good approximation for stress relaxation in a composite with delayed power law and non-zero impact response to a stress step, after an initial time interval and during a significant part (but not towards the end) of stress relaxation. We derive the same value for the parameter of the hyperbolic response and for the parameter of the stretched exponential, as well-defined function of the basic parameters of the viscoelastic hierarchy.

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[3] **The stretched exponential seems to have been discovered by R. Kohlrausch in 1847 while studying the relaxation of shear stress under conditions of constant shear strain, also in the silk threads of galvanometers.**

(2)-Second, to apply equation (5) to resonance experiments done with bars of power-law solids.

We separate variables in the case of a bar with free ends, assumed supported by its midpoint.

We derive approximate formulae for resonance frequencies.

Mechanical resonance or free vibrations produced in bodies of suitable shapes, sizes and material allows very accurate measurements of Young's elastic moduli. This is the case of concrete, glass, ceramics and metallic alloys. Longitudinal, flexural and torsion vibrations may be excited and measured in order to estimate the elasticity moduli (see, for example, Trianón and Griego, 1983 and the bibliography suggested in this report).

The following formula is used to estimate Young's modulus from the measurement of the fundamental frequency of resonance $f_1$ of longitudinal vibrations, the density $\rho$ of the material and a correction factor $C_L$ :

$$E_0 = \rho\ C_L\ f_1^2 \qquad (9)$$

For power-law solids, we suggest a way to incorporate the effect of the internal friction in the above mentioned correction factor.

## MATHEMATICAL MODELING

### (A)-Power-Law Solids

Let us consider a biphasic composite material. One phase is a linear viscous matrix. The other phase is a collection of linear elastic aggregates composed by many elements of different sizes and shapes.

We apply a normal stress $\sigma_1 = \sigma$ on the opposite faces of a volume element. Let $\varepsilon_1 = \varepsilon$ be the resulting strain in the direction of the applied stress. (Figure 1)

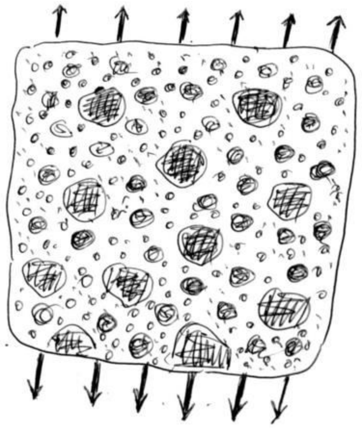

Figure 1.Element of volume of a biphasic composite with hierarchical structure.

Working in the framework of the elementary theory of composite materials, we suppose that the connection between stress and strain can be obtained in one dimension, for this composite, using a sequence of nested viscoelastic boxes (Figure 2).

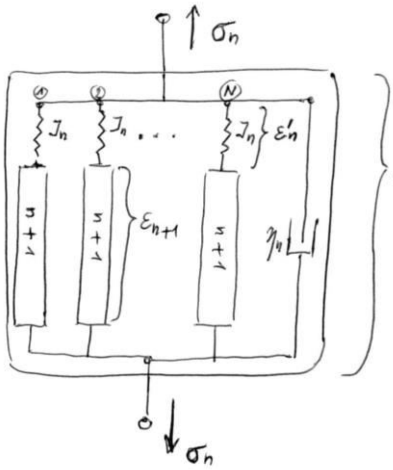

Figure 2.Viscoelastic behavior of the composite represented by a set of nested viscoelastic boxes.

The viscoelastic box of order n is composed of $N_n+1$elements in parallel: one linear viscous element with damping parameter $\eta_n$ and $N_n$ complex viscoelastic elements. Each of these complex elements is the in-series combination of a linear elastic element with compliance $J_n$ and a viscoelastic box of order n+1.

Then, if $\sigma_n(t)$ is the stress applied to the viscoelastic box of order n at time $t$, and if $\varepsilon_n(t)$ is its corresponding strain response, we have:

$$\varepsilon_n(t)=\int_0^t Q_n(t-t')\cdot\sigma_n(t')\cdot dt' \qquad (10)$$

We assume that the stress and strain are always zero when t<0. The creep kernel of the viscoelastic boxes of the level n in the hierarchical viscoelastic structure is $Q_n(t)$.

The viscoelastic box of order 1, which contains all the other boxes nested in its interior, is supposed to be in series with a single elastic element of compliance $J_0$. This element gives the impact or immediate elastic response of the considered element of volume of the composite material. If $\varepsilon'_n(t)$ is the strain produced in the elastic element of compliance $J_n$, then:

$$\varepsilon_n(t)=\varepsilon'_n(t)+\varepsilon_{n+1}(t) \qquad (11)$$

But $\varepsilon'_n(t)=J_n\cdot\sigma_{n+1}(t)$, so that:

$$\varepsilon_n(t)=J_n\cdot\sigma_{n+1}(t)+\int_0^t Q_{n+1}(t-t')\cdot\sigma_{n+1}(t')\cdot dt' \qquad (12)$$

But in parallel to the combination of elastic compliance with viscoelastic box of order n there is a viscous element.

The stress acting through it verifies:

$$\sigma_{nv}(t)=\eta_n\cdot\frac{\partial}{\partial t}\varepsilon_n(t) \qquad (13)$$

If $\varphi_{nv}$ is the volume fraction of this viscous element, referred to the whole volume of box number n, and if $\varphi_n$ is the volume fraction of the in series combination of an elastic element with box number n+1, from the elementary theory of composite materials we have, applying the popular rule of mixtures:

$$\sigma_n(t)=N_n\cdot\varphi_n\cdot\sigma_{n+1}(t)+\varphi_{nv}\cdot\sigma_{nv}(t) \qquad (14)$$

The volume fractions verify, for every n:

$$N_n\cdot\varphi_n+\varphi_{nv}=1 \qquad (15)$$

The possibility of applying the rule of mixtures, at least as a convenient approximation, can be justified as follows. If $A_n$ is a cross sectional area corresponding to the boxes of level n in the viscoelastic hierarchy and $A_{nv}$ is the area corresponding to the dashpot, then: $A_n \cdot \sigma_n(t) = N_n \cdot A_{n+1} \cdot \sigma_{n+1}(t) + A_{nv} \cdot \sigma_{nv}(t)$ so $\sigma_n(t) = N_n \cdot \frac{A_{n+1}}{A_n} \cdot \sigma_{n+1}(t) + \frac{A_{nv}}{A_n} \cdot \sigma_{nv}(t)$

As $A_n = N_n \cdot A_{n+1} + A_{nv}$, so we have $N_n \cdot \frac{A_{n+1}}{A_n} + \frac{A_{nv}}{A_n} = 1$

However, the average fraction of the area occupied by a certain component in a set of parallel cross sections of a composite body can be used as an estimation of the volume fraction of the component in the body (Delesse's rule; Weibel and Elias, 1967).

As consequence of this connection between area averages and volumes of a given phase in a body of composite material, if $\frac{A_{n+1}}{A_n}$ is interpreted as an estimation of the average fraction of parallel cross sectional areas of viscoelastic boxes of level n+1 inside a box of level, taking into account Delesse's rule, we have $\varphi_n \cong \frac{A_{n+1}}{A_n}$

An analogous derivation gives $\varphi_{nv} \cong \frac{A_{nv}}{A_n}$ Of course both stresses and strains must be interpreted also as suitable averages.

## (A.1)-Derivation of a functional equation for the creep kernel $Q_1(t)$

Taking Laplace's transforms, and defining $\bar{f}(s) = \int_0^{+\infty} e^{-s \cdot t} \cdot f(t) \cdot dt$, it follows from (11), (12), (13) and (14):

$$\bar{\varepsilon}_n(s) = \bar{Q}_n(s) \cdot \bar{\sigma}_n(s) \tag{16}$$

$$\bar{\sigma}_{nv}(s) = s \cdot \eta_n \cdot \bar{\varepsilon}_n(s) \tag{17}$$

$$\bar{\varepsilon}_n(s) = J_n \cdot \bar{\sigma}_{n+1}(s) + \bar{Q}_{n+1}(s) \cdot \bar{\sigma}_{n+1}(s) \tag{18}$$

$$\bar{\sigma}_n(s) = N_n \cdot \varphi_n \cdot \bar{\sigma}_{n+1}(s) + \varphi_{nv} \cdot \bar{\sigma}_{nv}(s) \tag{19}$$

Eliminating $\bar{\sigma}_n(s)$, $\bar{\sigma}_{n+1}(s)$ and $\bar{\sigma}_{nv}(s)$ between (7), (8), (9) and (10) we obtain:

$$\bar{Q}_n(s) = \cfrac{1}{\varphi_{nv} \cdot \eta_n \cdot s + \cfrac{N_n \cdot \varphi_n}{J_n + \bar{Q}_{n+1}(s)}} \tag{20}$$

Now, if we assume an infinite hierarchy of nested viscoelastic boxes, then from (20) we obtain the following continued fraction:

$$\bar{Q}_1(s) = \cfrac{1}{\varphi_{1v} \cdot \eta_1 \cdot s + \cfrac{N_1 \cdot \varphi_1}{J_1 + \cfrac{1}{\varphi_{2v} \cdot \eta_2 \cdot s + \cfrac{N_2 \cdot \varphi_2}{J_2 + \ldots}}}} \tag{21}$$

Now, let us suppose that: $N_n \cong N$ $J_n \cong J$ $\varphi_n \cong \varphi$ $\varphi_{nv} \cong \varphi_{nv}$ for every n.

Then (15) reduces to

$$N \cdot \varphi + \varphi_v = 1 \tag{22}$$

If the viscous dissipation of elastic energy is produced mainly in the region of matrix surrounding each aggregate-matrix interface, we can expect that as the aggregate becomes smaller, the viscosity becomes larger (the other conditions remaining unchanged).

So, introducing a parameter $\beta$ between 0 and 1, we assume that the viscosity coefficients of the different levels of the viscoelastic hierarchy verify:

$$\eta_n \cdot \left(\phi^{\beta}\right)^{n-1} \cong \eta_1 \tag{23}$$

So, as the volume fraction $\varphi$ is less than 1, $\eta_n$ grows scaled by $\frac{1}{\varphi^{\beta}}$ from each level to the following level of the hierarchy. Then (21) reduces to:

$$\overline{Q}_1(s) = \cfrac{1}{\varphi_v \cdot \eta_1 \cdot s + \cfrac{N \cdot \varphi}{J + \cfrac{1}{\varphi_v \cdot \cfrac{\eta_1}{\varphi^{\beta}} \cdot s + \cfrac{N \cdot \varphi}{J + \cfrac{1}{\varphi_v \cdot \cfrac{\eta_1}{\left(\varphi^{\beta}\right)^2} \cdot s + \cfrac{N \cdot \varphi}{J + \ldots}}}}}} \tag{24}$$

Taking (14) into account, and splitting the volume fraction $\varphi = \varphi^{\beta} \cdot \varphi^{1-\beta}$, it is possible to re-cast the continued fraction (24) as follows:

$$\overline{Q}_1(s) = \cfrac{1}{\varphi_v \cdot \eta_1 \cdot s + \cfrac{N \cdot \varphi^{1-\beta}}{\cfrac{J}{\varphi^{\beta}} + \cfrac{1}{\varphi_v \cdot \eta_1 \cdot s + \cfrac{N \cdot \varphi^{1-\beta}}{\cfrac{J}{\left(\varphi^{\beta}\right)^2} + \cfrac{1}{\varphi_v \cdot \eta_1 \cdot s + \ldots}}}}} \tag{25}$$

This expression is convergent because $N \cdot \varphi < 1$ (Wall, 1972).

For a given set of parameters of the viscoelastic hierarchy, (16) defines a function of $s$ for positive values of the Laplace's variable.

If we substitute in (25) $s$ by $\varphi^{\beta} \cdot s$ we obtain, after some operations, the following *functional equation:*

$$\frac{\overline{Q}_1\left(\varphi^{\beta} \cdot s\right)}{J} = \cfrac{1}{\varphi^{\beta} \cdot \left( T \cdot s + \cfrac{N \cdot \varphi^{1-\beta}}{1 + \cfrac{\overline{Q}_1(s)}{J}} \right)} \tag{26}$$

The characteristic time $T$ is defined by the equation $T = \varphi_v \cdot \eta_1 \cdot J$ (27)

Equation (26) can be solved numerically. However, here we want to proceed analytically as far as possible, so we are going to derive an asymptotic functional equation whose analytic solution is known. As also shown in the mathematical Appendix, assuming that $N \cdot \varphi^{1-\beta} > 1$, when $s \downarrow 0$ then $\overline{Q}_1(s) \uparrow \infty$ and $s \cdot \overline{Q}_1(s) \downarrow 0$ As consequence, when $T \cdot s$ is small enough relative to 1 we will have $\frac{s \cdot \overline{Q}_1(s)}{J}$ small relative to 1 and $\frac{\overline{Q}_1(s)}{J}$ large relative to 1, so (26) may be approximated by:

$$\bar{Q}_1\left(\varphi^{\beta}\cdot s\right)=\frac{1}{N\cdot\varphi}\cdot\bar{Q}_1(s) \qquad (28)$$

Regular solutions of this functional equation are of the following type:

$$\bar{Q}_1(s)=\frac{A}{s^{p}} \qquad (29)$$

The parameter $p$ is given as the following function of some parameters of the viscoelastic hierarchy:

$$p=\frac{\ln\left(\dfrac{1}{N\cdot\varphi}\right)}{\ln\left(\dfrac{1}{\varphi^{\beta}}\right)} \qquad (30)$$

As will be shown later, $p$ is a parameter comprised always between 0 and 1.

Inverting the Laplace transform, we obtain the desired delayed strain kernel that gives a power law strain response to a step of stress input:

$$Q_1(t)=Q_1(t_0)\cdot\left(\frac{t_0}{t}\right)^{\alpha} \qquad (31)$$

Here $\alpha=1-p$ (also comprised between 0 and 1), and $t_0$ is an arbitrary but positive reference time. If we take $t_0=T_{\varepsilon}$, we can expect that (31) approximates the viscoelastic strain kernel for $t>T_{\varepsilon}$. However, $C$ remains undetermined, as well as $Q_1(T_{\varepsilon})$ which is proportional to $C$.

### (A.2)-Derivation of the stretched exponential stress relaxation

If a linear viscoelastic material has both instantaneous (elastic) $\varepsilon_0(t)$ and delayed $\varepsilon_d(t)$ strain response to an applied stress $\sigma(t)$ that is zero for negative times, the total strain $\varepsilon_0(t)+\varepsilon_d(t)$ is given by:

$$\varepsilon(t)=J_0\left(\sigma(t)+\int_0^t K(t-t')\,\sigma(t')\,dt'\right) \qquad (32)$$

If the case of the composite material studied in this work, the creep kernel $K(t)$ may be identified with $Q_1(t)$, the kernel of the box in level one in the viscoelastic hierarchy.

Inverting (32) and introducing an elastic impact modulus $E_0$ such that $J_0\cdot E_0=1$ and a stress relaxation kernel $L(t)$:

$$\sigma(t)=E_0\left(\varepsilon(t)-\int_0^t L(t-t')\,\varepsilon(t')\,dt'\right) \qquad (33)$$

Taking Laplace's transforms in (32) and (33), it is possible to relate the transform $\bar{L}(s)$ of the stress relaxation kernel with the transform $\bar{K}(s)$ of the creep kernel:

$$\bar{L}(s)=\frac{\bar{K}(s)}{1+\bar{K}(s)} \qquad (34)$$

Now we substitute the Laplace's transform $\bar{Q}_1(s)=\frac{A}{s^{p}}$ of the power law kernel (see equation (29)) in place of $\bar{K}(s)$ in (34), with $\frac{A}{J_0}=\left(\frac{\chi}{T_{\varepsilon}}\right)^{p}$, and we obtain:

$$\bar{L}(s)=\frac{\left(\frac{\chi}{s\cdot T_{\varepsilon}}\right)^{p}}{1+\left(\frac{\chi}{s\cdot T_{\varepsilon}}\right)^{p}} \tag{35}$$

From $\frac{A}{J_0}=\left(\frac{\chi}{T_{\varepsilon}}\right)^{p}$ and from (34) it follows that: $\chi=\left(\frac{J}{J_0}\right)^{\frac{1}{p}}\cdot p^{\frac{1}{p}-2}\cdot\left(\frac{\Gamma(p)}{\Gamma(1-p)}\right)^{\frac{1}{p}}$ (36)

Inverting the transform (35):

$$L(t)=\frac{\chi}{T_{\varepsilon}}\cdot\left(\frac{T_{\varepsilon}}{\chi\cdot t}\right)^{1-p}\cdot\sum_{n=0}^{\infty}\frac{(-1)^{n}}{\Gamma\left((n+1)\cdot p\right)}\cdot\left(\left(\frac{\chi\cdot t}{T_{\varepsilon}}\right)^{p}\right)^{n} \tag{37}$$

As usual, $\Gamma(z)$ represent the Gamma function. An equivalent expression was called *fractional exponential function* by Rabotnov, already in 1948, since for $p=1$ reduces to the decaying ordinary exponential. In our case, when $p=1$ we obtain: $\left(\frac{\chi}{T_{\varepsilon}}\right)\cdot e^{-\frac{t}{\left(T_{\varepsilon}/\chi\right)}}$

The function $\sum_{n=0}^{\infty}\frac{(-1)^{n}}{\Gamma\left((n+1)\cdot p\right)}\cdot(u)^{p\cdot n}$ is approximated by (Rabotnov, 1980): $p\cdot(2-p)^{p}\cdot e^{-p\cdot(2-p)^{p}\cdot u^{p}}$ then:

$$L(t)\approx\frac{\chi}{T_{\varepsilon}}\cdot\left(\frac{T_{\varepsilon}}{\chi\cdot t}\right)^{1-p}\cdot p\cdot(2-p)^{p}\cdot\exp\left[-p\cdot(2-p)^{p}\cdot\left(\frac{\chi\cdot t}{T_{\varepsilon}}\right)^{p}\right] \tag{38}$$

The relaxation of stress at a constant strain $\varepsilon_0$ for a viscoelastic body is given by $\sigma(t)=M(t)\cdot\varepsilon_0$ with, by definition:

$$M(t)=E_0\left(1-\int_0^t L(t')\cdot dt'\right) \tag{39}$$

Taking into account (41), from (40) it follows that the stress relaxes according to a stretched exponential: $\sigma(t)=M(t)\cdot\varepsilon_0=\sigma_0\cdot e^{-\left(\frac{t}{T_{\sigma}}\right)^{p}}$ (40)

In this equation $\sigma_0=M_0\cdot\varepsilon_0$

From equations (38) and (40) we obtain the characteristic stress relaxation time $T_{\sigma}$ given as function of the characteristic time $\tau_{\varepsilon}$ of the hyperbolic strain response:

$$T_{\sigma}=\frac{T_{\varepsilon}}{(2-p)\cdot\sqrt[p]{p}\cdot p^{\frac{1}{p}-2}\cdot\left(\frac{\Gamma(p)}{\Gamma(1-p)}\right)^{\frac{1}{p}}\cdot\left(\frac{J}{J_0}\right)^{\frac{1}{p}}} \tag{41}$$

## (B)-Mechanical vibrations and resonance from the generalized Love's equation for power-law solids

Let us consider the viscoelastic generalization of Love's equation in absence of distributed external forces, so $X(t,x)$ is always zero.

Then:

$$\frac{\partial^2 s(t,x)}{\partial t^2} = c_0^2 \left( \frac{\partial^2 s(t,x)}{\partial x^2} - \int_{-\infty}^{t} L(t-t') \, \frac{\partial^2 s(t',x)}{\partial x^2} \, dt' \right) + (\nu_0 r_0)^2 \, \frac{\partial^4 s(t,x)}{\partial x^2 \, \partial^2 t} \tag{42}$$

Now, if the material behaves as a power-law solid, the kernel $L(t)$ is given by a fractional exponential function (37) that can be approximated by equation (38).

Let us consider a bar of length $\ell$, and put the origin of coordinates at one end, so $0 \le x \le \ell$.

If the bar has free ends, the longitudinal stresses verify $\sigma(t,0) = \sigma(t,\ell) = 0$ for each instant of time.

From equation (4 b) it follows that the longitudinal strains verify Neumann boundary conditions: $\varepsilon(t,0) = \frac{\partial}{\partial x} s(t,0) = 0 \quad \varepsilon(t,\ell) = \frac{\partial}{\partial x} s(t,\ell) = 0$ for every instant of time.

Separating variables by introducing the ansatz $s(t,x) = T(t)\,S(x)$ in equation (42), from the boundary conditions we obtain the normalized Eigen functions:

$$S(x) = S_m(x) = \sqrt{\frac{2}{\ell}} \, \cos(k_m x) \qquad k_m = \frac{m\pi}{\ell} \qquad m = 1,2,3,... \tag{43}$$

To each Eigen function it correspond a time function $T(t) = T_m(t)$ that verifies the integral-differential equation:

$$\frac{d^2}{dt^2} T_m(t) + \omega_{0,m}^2 \left( T_m(t) - \int_{-\infty}^{t} L(t-t') T_m(t') dt' \right) = 0 \tag{44 a}$$

In this equation:

$$\omega_{0,m}^2 = \frac{c_0^2 \, k_m^2}{1 + (\nu_0 r_0)^2 \, k_m^2} \tag{44 b}$$

If $T_m(t)$ is zero if $t < 0$:

$$\int_{-\infty}^{t} L(t-t') T_m(t') dt' = \int_{0}^{t} L(t-t') T_m(t') dt' = \int_{0}^{t} L(t') T_m(t-t') dt'$$

Then the equation (44 a) can be solved by the Laplace's transform in terms of the initial conditions $T_m(0^+)$ and $\frac{d}{dt} T_m(0^+)$ for $T_m(t)$.

If the internal friction of a power-law solid is low enough, the following approximate asymptotic formula for the fundamental frequency of free oscillation can be derived from (44 a), with $\alpha(p) = p \cdot (2-p)^p$:

$$\omega_1^2 \simeq \omega_{0,1}^2 \left( 1 - \int_0^{+\infty} \frac{\chi}{T_\varepsilon} \left( \frac{T_\varepsilon}{\chi t} \right)^{1-p} \alpha(p) \, \exp\left[ -\alpha(p) \left( \frac{\chi t}{T_\varepsilon} \right)^p \right] \cos(\omega_{0,1} t') \, dt' \right) \tag{45}$$

## DISCUSION

### (D.1)- The parameter $p$ of the power law strain response

From (32) we conclude that $p$ must be positive, because both $\xi = \frac{1}{N \cdot \varphi}$ and $\frac{1}{\varphi^{\beta}}$ are greater than 1.

But from both, experimental results and thermodynamic requirements related with elastic energy dissipation, we know that $p$ must be less than 1.

This is equivalent to the restriction $v > 1$. Taking into account the definition of $v$ given in equation (29), as well as the inequality $N \cdot \varphi < 1$, we see that $v > 1$ is verified if and only if $N \cdot \varphi^{1-\beta} > 1$.

The physical origin of this last inequality and the physical origin of the scale law for the dashpot viscous moduli $\eta_n \cdot \left(\varphi^{\beta}\right)^{n-1} \cong \eta_1$, can be found examining the geometric scaling laws of the viscoelastic hierarchy.

Let us consider an element of volume $V_n$ in the composite material corresponding to a combination in series of an elastic element and a whole viscoelastic box of level n.

Let us introduce a characteristic length $l_n$ corresponding to volume $V_n$ and a characteristic area $S_n \approx l_n^2$ of the interfaces between the volume element and its viscous matrix environment. We assume that $l_{n+1} \approx \varphi^{1/3} \cdot l_n$, so $S_{n+1} \approx \varphi^{2/3} \cdot S_n$ and $V_{n+1} \approx \varphi \cdot V_n$

Furthermore, we suppose that the force $F_n$ taken by the viscous matrix in the volume element is given by $F_n = \mu \cdot S_n \cdot \frac{\partial \varepsilon_n}{\partial t}$

As consequence, the force per unit volume $\hat{\sigma}_{nv}$ is given by:

$$\hat{\sigma}_{nv} = \frac{F_n}{V_n} = \mu \cdot \frac{S_n}{V_n} \cdot \frac{\partial \varepsilon_n}{\partial t} = \left( \varphi^{-\frac{1}{3}} \right)^{n-1} \cdot \mu \cdot \frac{S_1}{V_1} \cdot \frac{\partial \varepsilon_n}{\partial t}$$

We suppose now that the viscous stress $\sigma_{nv}$ is related with the density of volumetric viscous forces by a constant factor: $\sigma_{nv} = \gamma \cdot \hat{\sigma}_{nv}$ So that $\sigma_{nv} = \eta_n \cdot \frac{\partial \varepsilon_n}{\partial t}$, with $\eta_n = \left( \varphi^{-\frac{1}{3}} \right)^{n-1} \cdot \eta_1$ and $\eta_1 = \gamma \cdot \mu \cdot \frac{S_1}{V_1}$

As consequence, we obtain the scale law of viscous dissipation $\eta_n \cdot \left(\varphi^{\beta}\right)^{n-1} \cong \eta_1$ with $\beta = \frac{1}{3}$ According to this model, the parameter of the power law and of the stretched exponential verifies $p = \frac{\ln\left(\frac{1}{N \cdot \varphi}\right)}{\ln\left(1 / \varphi^{\frac{1}{3}}\right)}$.

Now $v$ is positive and $p$ is less than one if $N \cdot S_{n+1} \big/ S_n \approx N \cdot \varphi^{2/3} = \left(N \cdot \varphi^{1-\beta}\right)$ is greater than 1: this happens if the total area related with viscous dissipation increases from one level of the hierarchy to the next one.

### (D.2)-The relaxation times in a self-similar viscoelastic hierarchy

The self-similar nature in physical space of the viscoelastic hierarchy proposed here gives rise to power law responses, and power law responses point to scale invariance in time. Scale invariance, at its turn, implies a manifold of time scales in the delayed strain responses $\varepsilon_d(t) = J(1) \cdot t^p \cdot \sigma_0$, such that the local time scale $\dfrac{|\varepsilon_d(t)|}{|d\varepsilon_d(t)/dt|}$ of the response at the instant $t$ is always proportional to $t$: $\dfrac{|\varepsilon_d(t)|}{|d\varepsilon_d(t)/dt|} = \dfrac{t}{p}$.

The characteristic times $\tau_\varepsilon$ do not appear in the local time scales.

In the case of stretched exponential stress relaxation $\sigma_r(t) \approx \sigma(0) \cdot \exp\left[-\left( t \big/ T_\sigma \right)^p\right]$ the local time scale is: $\dfrac{|\sigma_r(t)|}{|d\sigma_r(t)/dt|} = \left( T_\sigma \big/ t \right)^p \cdot t$

It grows like $t^{1-p}$, slower than the growth of the local time scale of the delayed strain response. But now the characteristic time $T_\sigma$ appears in the formula for the local time scale: the stretched exponential does not have scale invariance in time.

### (D.3)-Correction factor to estimate Young's modulus

When the masses of transducers used to excite and register longitudinal vibrations in a bar are negligible, and internal friction is negligible, formula (45) reduces to

$$f_1^2 = f_{0,1}^2 = \left(\frac{1}{4\pi^2}\right)\left(\frac{c_0^2\, k_m^2}{1 + \left(v_0\, r_0\right)^2 k_m^2}\right) \tag{46}$$

From equation (6), it follows that Young's modulus Young's modulus $E_0$ is related to the density and the velocity of extensional waves: $E_0 = \rho c_0^2$

If Poisson's modulus is known for the material of the bar, Young's modulus can be estimated from the fundamental frequency using equations (9) and (46):

$$E_0 = \rho\, C_L\, f_{0,1}^2 \qquad\qquad C_L = 4\ell^2\left(1 + \frac{\left(v_0\, r_0\, \pi\right)^2}{\ell^2}\right)$$

If internal friction must be considered, from formula (45) we derive $E_0 = \rho\, C_L\, f_1^2$ , but now with $C_L \approx \dfrac{4\ell^2}{\left(1 - \int_0^{+\infty} L(t') \cos\left(2\pi f_{0,1}\, t'\right)\, dt'\right)}\left(1 + \dfrac{\left(v_0\, r_0\, \pi\right)^2}{\ell^2}\right)$

Through $f_{0,1}$, Young's modulus appears in the integrand.

So the formula above could be applied to relate Young's modulus $E_0$ with the measured fundamental frequency in resonance only if a good enough preliminary estimation of the modulus $E_0$ is available to calculate an approximation to $f_{0,1}$.

It would seem that the analytical approach, as applied in this work, should be abandoned.

## CONCLUSIONS

(a) An analytical formula for the power law parameter as function of the branching number and volume fractions of the hierarchical composite material was derived for a simplified example. The stretched exponential was obtained as an approximation to stress relaxation when strain creeps according to a power-law. In this approximation the parameter of the stretched exponential turned to be the same as the power law parameter.

(b) The functional equation method does not allow us to derive an analytical formula to relate the constant $C$ that appears in the creep formula $\varepsilon_d(t) \approx C \cdot \left(\frac{t}{T_\varepsilon}\right)^p \cdot \sigma_0$ with the parameters of the viscoelastic hierarchy. This poses a new research problem.

(c) A formula was obtained that relates the characteristic times $T_\varepsilon$ and $T_\sigma$ with the mechanical parameters of the composite material.

(d) Due to its scale invariance, the hyperbolic strain response has an infinite and unbounded set of local time scales $\frac{t}{p}$. The characteristic time $T_\varepsilon$ must not be considered as a time scale of the hyperbolic response. It gives a measure of the numerical order of the interval of time (since the application of a step in stress) after which the power law can be applied.

(e) The characteristic time $T_\sigma$ gives a measure of the numerical order of the interval of time (since the application of a step in strain) after which the stretched exponential law for stress relaxation can be applied. The stretched exponential also has an infinite and unbounded set of local time scales $\left({}^{T_\sigma}\!/_{t}\right)^p \cdot t$ Contrary to the hyperbolic response case, now $T_\sigma$ appears in formula for the local time scale of stress relaxation.

(f) In principle the equation (12) for the Laplace's transform of the creep kernel allows to take into account random branching numbers as well as random elastic and viscous moduli. The resulting equation could be approximately solved applying a method developed by Kaplan and Gray to study the effect of disorder on a fractal model for the alternating current response of rough interfaces (Kaplan and Gray, 1985).

(g) Finite viscoelastic hierarchies show, besides the high frequency cut off that corresponds to the beginning of the deformation process, a low frequency cut off for long times of deformation. In the model based in a set of nested boxes, the low frequency cut off behaviour depends of the termination (elastic elements or dashpots) of the last box in the hierarchy.

(h) From the difficulties found with the analytical approach to mechanical vibrations intended here, it seems that a fully numerical approach to the generalization of Love's equation for power-law solids must be developed.

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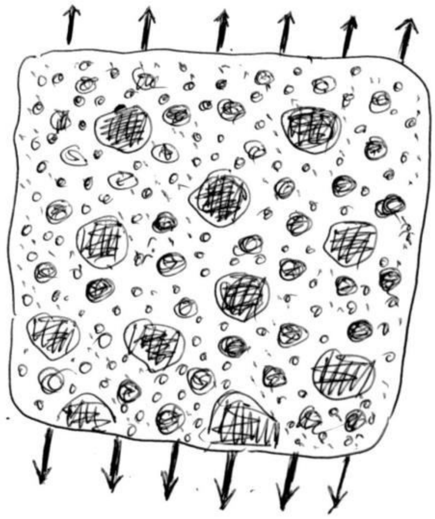
*Dr. S. H. Liu* *February 26, 1988*
*Solid State Division*
*Oak Ridge National Laboratory*
*Oak Ridge*
*Tennessee 37831* *USA*

Dear Dr. Liu[1]:

Thank you for your kind letter of February 4, 1988; for your letter of November 13, 1986, and for the reprints on your theory about the ac response of rough interfaces!
It seems to me that my letter of November 24, 1987 is lost. So I will reproduce here what I wrote there.

Certain of your results about rough interfaces were applied here, in Uruguay, to improve the design of the tip of a stimulating electrode of complex shape which is now being used for medical purposes.
While I was studying the reprints I perceived a close connection between your explanation of constant phase angle impedance and a possible explanation of Hopkinson's power law for viscoelastic solids.

Consider a biphasic composite material.
One phase is (mainly) a linear viscous matrix and the other phase is (mainly) a linear elastic aggregate. The aggregate is composed of many elements of different sizes and (perhaps) shapes.



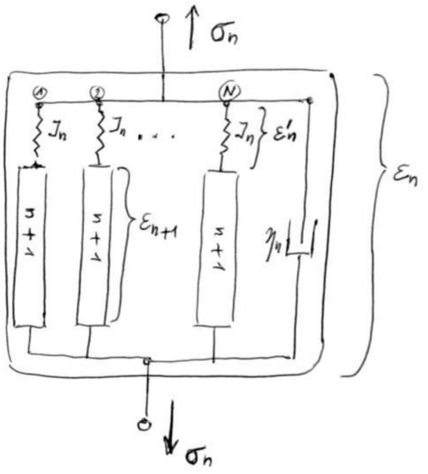
Let us suppose that we apply a stress $\sigma_1 = \sigma$ on opposite faces of a volume element of the material.

---

**[1] The original letter was hand-written. The pictures that appear here are computer scans of the original drawings. The fractal model explained in this letter was previously part of the presentation by R. Suárez Antola, "Vibrations of viscoelastic, power-law solids", in the First Regional Seminar of Latinoamerican and Caribbean on Vibration Analysis and Diagnostics, Bariloche's Atomic Center, Bariloche, Argentina, September 14 to 18, 1987.**

Let $\varepsilon_1 = \varepsilon$ be the resulting strain in the direction of applied stress.

Working in the frame of elementary theory of composite materials, we suppose that the connection between stress and strain in one dimension can be obtained using a sequence of nested viscoelastic "boxes" as shown in the following picture:



Viscoelastic "box" of order $n$ is composed by $N_n+1$ elements in parallel: one linear viscous element with "viscosity" $\eta_n$ and $N_n$ complex viscoelastic elements.

Each of these elements is a combination, in series, of a linear elastic element with elastic compliance $J_n$ and a viscoelastic "box" of order $n+1$.

If $\sigma_n(t)$ is stress applied to viscoelastic box of order $n$ at time $t$ and if $\varepsilon_n(t)$ is its corresponding strain, we have: $\varepsilon_n(t) = \int_0^t Q_n(t-\tau)\sigma_n(\tau)d\tau$ [1]

If $\varphi_{nv}$ is the volume fraction (referred to the whole volume of box number $n$) of the viscous element in parallel, and if $\varphi_n$ is the volume fraction of the in-series combination of an elastic element and a box of order $n+1$, we suppose that:

$$\sigma_n(t) = N_n\,\varphi_n\,\sigma_{n+1}(t) + \varphi_{nv}\,\sigma_{nv}(t) \qquad [2]$$

Here $\sigma_{nv}(t)$ is stress applied to the viscous element, and $N_n\,\varphi_n + \varphi_{nv} = 1$ [3]

Besides:
$$\varepsilon_n(t) = J_n\,\sigma_{n+1}(t) + \int_0^t Q_{n+1}(t-\tau)\sigma_{n+1}(\tau)d\tau \qquad [4]$$

And
$$\sigma_{nv}(t) = \eta_n \frac{\partial}{\partial t}\varepsilon_n(t) \qquad [5]$$

Taking Laplace's transform in equations [1], [2], [4] and [5], after some work we obtain ($s$ is Laplace's variable):

$$\bar{\varepsilon}_n(s) = \bar{Q}_n(s)\,\bar{\sigma}_n(s) \;; \qquad \bar{Q}_n(s) = \frac{1}{\varphi_{nv}\,\eta_n\,s + \dfrac{N_n\,\varphi_n}{J_n + \bar{Q}_{n+1}(s)}} \qquad [6]$$

In order to simplify the analysis, from now on we shall take $N_n = N$, $J_n = J$, $\varphi_n = \varphi$, $\varphi_{nv} = \varphi_v$ for every $n$.

But if viscous dissipation of elastic energy is related mainly with the region of matrix surrounding each aggregate-matrix interface, we expect that, as aggregate becomes smaller, viscosity becomes larger (the other things remaining unchanged).

If we suppose that: $$\eta_n = \left(\frac{1}{\varphi}\right)^n \eta \qquad (n = 1, 2, 3, \ldots) \qquad [7]$$

It is possible to show, beginning with equation [6] and taking $n = 1$:

$$\bar{Q}_1(\varphi s) = \frac{1}{\varphi}\left\{\frac{1}{(\varphi_v \eta J)s + \dfrac{N}{1 + \dfrac{\bar{Q}_1(s)}{J}}}\right\} \qquad [8]$$

Now, let $T_c = \varphi_v \eta J$ be a characteristic time of the material.

Then, it $T_c s \ll 1$, $\dfrac{s\bar{Q}_1(s)}{J} \ll 1$ and $\dfrac{\bar{Q}_1(s)}{J} \gg 1$, we obtain the same basic functional equation of your theory of a.c. response of rough interfaces:

$$\bar{Q}_1(\varphi s) \cong \frac{1}{\varphi N}\bar{Q}_1(s) \qquad [9]$$

Proceeding as you have done, we find the solution $$\bar{Q}_1(s) = \frac{A}{s^p} \qquad [10]$$

Where $$p = 1 - \frac{\log_e N}{\log_e\left(\dfrac{1}{\varphi}\right)} \qquad [11]$$

As, by equation [3], $0 < N\varphi < 1$, we always have $N < \dfrac{1}{\varphi}$. Then $0 < p \le 1$ Besides, if $N > 1$ then $p < 1$.

Applying a Heavside's step stress history $\sigma(t) = \left\{\begin{matrix} 0 \ if \ t < 0 \\ \sigma_0 \ if \ t > 0 \end{matrix}\right\}$ we have:

$$\bar{\varepsilon}_1(s) = \bar{Q}_1(s)\,\frac{\sigma_0}{s} = \frac{A}{s^{p+1}}\sigma_0$$

Inverting the transform we find the well-known version of Hopkinson's power law:

$\varepsilon(t) = \left(\dfrac{A}{\Gamma(p+1)}\right) t^p \, \sigma_0$ where $\Gamma(z)$ is Euler's gamma function, and $0 < p \le 1$.

Experimental values of $p$ are in fact bounded between 0 and 1. But now $p$, instead of being an empirical parameter is expressed, by equation [11], as a definite function of branching number $N$ and scaling parameter $\dfrac{1}{\varphi} = \dfrac{N}{1-\varphi_v}$.

It seems to me that this approach can be generalized in at least two complementary directions:

First, it can be randomized as your theory was in the paper of Kaplan and Gray.

Second, it can be extended to a nested hierarchy of viscoelastic fractals with varying fractal parameters, so that each one has its own $p$. In this case, as time grows (or as frequency lowers), "equivalent $p$" will be modified as is actually seen during certain experiments with polymers and (perhaps) with concrete?

As far as I know, power-law models are used in experimental work with viscoelastic materials because they are simple and fairly accurate, and in theoretical work because they allow us to use the well-developed mathematical framework of Abel's and fractional exponential operators.

But if this approach is correct, it seems that it can be used to design viscoelastic materials with pre-stablished $p$ values and to explain experimental variations in the exponent of Hopkinson's power-law in terms of structural variations in the material.

Perhaps this kind of approach can be applied to polycrystalline materials under suitable conditions, assigning elastic properties to the bulk of the grains and viscous properties to the interfaces between grains.

I must confess that in my personal situation I have certain difficulties to get information about what is happening in outside world. So, perhaps someone else has done some work in this same direction and I don't know.

Anyhow, I can't develop the experimental aspects of this approach in my country.

If it is possible, I would like to know if you have published further work on your theory of rough interfaces and related topics.

It seems to me that this approach in temporal domain to viscoelastic power-law suggests a natural generalization: Substitute stress $\sigma$ for any suitable spatially intensive variable ("pervariable" in the terminology of systems engineering), substitute strain $\varepsilon$ for the corresponding conjugated (in relation with energy) spatially extensive variable ("transvariable") and substitute the elastic and viscous laws, $\varepsilon = J\ \sigma$ and $\sigma = \eta \frac{\partial}{\partial t}\varepsilon$ by homologous laws in terms of the new variables. In this case a hierarchy of nested boxes constructed in the same fashion used to obtain Hopkinson's power-law, with suitable branching numbers and scaling parameters, will end in a relation between transvariable and pervariable which is hyperbolical for times long enough or frequencies low enough. Perhaps we have here a possible approach to "Joseph effect" (as defined by Mandelbrot)?

Yours truly

Roberto Suárez-Antola

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April 18, 1988

Dr. Roberto Suarez Antola
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Montevideo, Uruguay

Dear Dr. Suarez Antola:

Thank you for your letter of February 26 which allowed me to understand your interesting idea on the viscoelastic properties of composite solids. I have done a limited amount of literature search, and as far as I can tell, no one has attempted what you have done. Therefore, I strongly encourage you to write it up and submit it for publication in the Physical Review. Also, if you can find the means to support you, I think you should present your work at the next Materials Research Society meeting to be held in Boston later this year. I will send you the information when I receive it.

As to the work itself, I think it makes eminent sense. Just like my own paper, it relies on the initial assumption that there is a hierarchy of structure. Purists may object to this assumption, as they did when my paper was first submitted for publication. Now the fractal nature of surfaces has been firmly established. There exists some evidence of hierarchical structure in composite materials, notably clay and rock. You may cite these as motivation for proposing such a model. The last paragraph of your letter touches upon a problem I have been studying, i.e. the dielectric response of fast ion conductors, which consist of a random liquid of ions moving in a regular crystalline lattice. Again, a hierarchical structure of the ionic distribution will establish the observed power-law frequency dependence, but strong evidence for such a distribution is lacking.

I am also enclosing a recent review of our work on the interfacial problem. This represents our latest thinking on this subject.

Sincerely yours,

S. H. Liu

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Solid State Division

SHL:vh

Enclosure

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# Power Law and Stretched Exponential Responses in Composite Solids

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**Keywords:** Solid composites; Elastic after-effect; Fractal hierarchy; Power law strain; Stretched exponential stress relaxation; Fractance; Mathematical modelling.

**Abstract.** Clay, rocks, concrete and other composite solids show evidence of a hierarchical structure. A fractal tree of nested viscoelastic boxes is proposed to describe the elastic after-effects in these composite solids. A generalized fractal transmission line approach is developed to relate the strain and stress responses. Power law for strain, under an applied stress step, is derived. The exponent in the power law is obtained as a well-defined function of the branching numbers and scaling parameters of the viscoelastic hierarchy. Then, a composite solid with both instantaneous (linear) elastic strain response and power law type (linear) elastic after-effect for an applied stress step, is considered. The stretched exponential stress relaxation to an applied strain step is derived as an approximation. For the same composite solid, the stretch parameter of the stretched exponential and the exponent of the power law result to be equal to each other.

## Introduction

Certain composite solids, that show evidence of having a hierarchical structure, like clays, some rocks, concrete and biological materials, in the linear regime, often present hyperbolic delayed strain responses (known as power law creep) to a step of stress, and exponentially stretched stress responses to a step of strain, during long periods of time, after a suitable initial time interval ([1] to [7]).

When a constant stress of magnitude $\sigma_0$ is applied at the origin of the time variable $t$, the hyperbolic law for delayed strain is $\varepsilon_d(t) \approx K \cdot \left(\frac{t}{\tau_\varepsilon}\right)^p \cdot \sigma_0$ being $K$ a positive parameter. This power law fits very accurately experimental data for times $t$ greater than the characteristic time $\tau_\varepsilon$. The parameter $p$ is always between 0 and 1.

For a variable applied stress $\sigma(t)$ the delayed strain response of a linear material with power-law viscoelasticity, is given by $\varepsilon_d(t) = \int_0^t Q(t-t') \cdot \sigma(t') \cdot dt'$. The viscoelastic kernel is: $Q(t) = Q(\tau_\varepsilon) \cdot \left(\frac{\tau_\varepsilon}{t}\right)^\alpha$ $\alpha = 1 - p$ $Q(\tau_\varepsilon) = \frac{(1-\alpha) \cdot K}{\tau_\varepsilon}$ The stretched exponential stress relaxation after a step of strain is given by $\sigma_r(t) \approx \sigma(0) \cdot \exp\left[-\left(\frac{t}{\tau_\sigma}\right)^q\right]$. For times $t$ greater than $\tau_\sigma$, the stretched exponential often fits to experimental data fairly accurately. The parameter $q$ also is always between 0 and 1.

A hierarchical and fractal viscoelastic model, to describe the mechanical responses of certain composite solids, was recently proposed [8]. A functional equation for the Laplace's transform of the viscoelastic strain kernel was derived, giving the desired asymptotic power law strain. For solids with elastic impact responses a delayed stretched exponential stress relaxations was obtained. However an indeterminate constant appears in the solution of the functional equation for the power law strain and reappears in the stretched exponential stress relaxation.

The purposes of the present paper are threefold: 1-To review briefly the above mentioned fractal viscoelastic model of composite solid. 2-To apply a generalized fractal transmission line approach (initially presented in [8]) instead of the functional equation approach detailed in [8]. 3-To derive the power law and stretched exponential responses without the appearance of undetermined parameters.

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## Mathematical Modeling of the Viscoelastic Hierarchy

Let us consider a biphasic composite material. One phase is a linear viscous matrix. The other phase is a collection of linear elastic aggregates composed by many elements of different sizes and shapes.

We apply a normal stress $\sigma_1 = \sigma$ on the opposite faces of a volume element. Let $\varepsilon_1 = \varepsilon$ be the resulting strain in the direction of the applied stress (Fig.1 left).

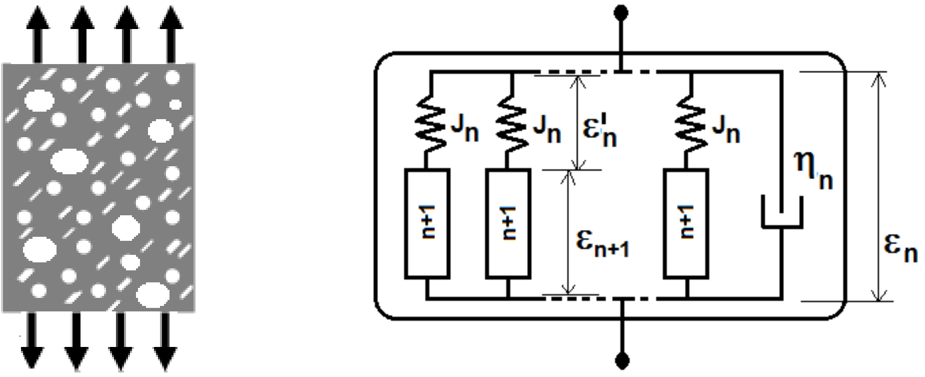

Fig.1: Left: Element of volume of a biphasic composite with hierarchical structure. Right: viscoelastic behavior of the composite represented by a set of nested viscoelastic boxes.

Working in the framework of the elementary theory of composite materials, we suppose that the connection between stress and strain can be obtained in one dimension, for this composite, using a sequence of nested viscoelastic boxes (Figure 1, right). The viscoelastic box of order n is composed of $N_n$+1elements in parallel: one linear viscous element with damping parameter $\eta_n$ and $N_n$ complex viscoelastic elements. Each of these complex elements is the in series combination of a linear elastic element with compliance $J_n$ and a viscoelastic box of order n+1. Then, if $\sigma_n(t)$ is the stress applied to the viscoelastic box of order n at time $t$, and if $\varepsilon_n(t)$ is its corresponding strain response, we have:

$$\varepsilon_n(t) = \int_0^t Q_n(t-t')\cdot\sigma_n(t')\cdot dt' \,. \tag{1}$$

We assume that the stress and strain are always zero when t<0. The creep kernel of the viscoelastic boxes of the level n in the hierarchical viscoelastic structure is $Q_n(t)$. The viscoelastic box of order 1, which contains all the other boxes nested in its interior, is supposed to be in series with a single elastic element of compliance $J_0$. This element gives the impact or immediate elastic response of the considered element of volume of the composite material. If $\varepsilon'_n(t)$ is the strain produced in the elastic element of compliance $J_n$, then:

$$\varepsilon_n(t) = \varepsilon'_n(t) + \varepsilon_{n+1}(t) \,. \tag{2}$$

But $\varepsilon'_n(t) = J_n \cdot \sigma_{n+1}(t)$, so that:

$$\varepsilon_n(t) = J_n \cdot \sigma_{n+1}(t) + \int_0^t Q_{n+1}(t-t')\cdot\sigma_{n+1}(t')\cdot dt' \,. \tag{3}$$

But in parallel to the combination of elastic compliance with viscoelastic box of order n there is a viscous element. The stress acting through it verifies:

$$\sigma_{nv}(t) = \eta_n \cdot \frac{\partial}{\partial t}\varepsilon_n(t) \,. \tag{4}$$

If $\varphi_{nv}$ is the volume fraction of this viscous element, referred to the whole volume of box number n, and if $\varphi_n$ is the volume fraction of the in series combination of an elastic element with box number n+1, from the elementary theory of composite materials we have, applying the popular rule of mixtures:

$$\sigma_n(t)=N_n\cdot\varphi_n\cdot\sigma_{n+1}(t)+\varphi_{nv}\cdot\sigma_{nv}(t). \tag{5}$$

The volume fractions verify, for every n:

$$N_n\cdot\varphi_n+\varphi_{nv}=1. \tag{6}$$

The possibility of applying the rule of mixtures, at least as a convenient approximation, can be justified as follows. If $A_n$ is a cross sectional area corresponding to the boxes of level n in the viscoelastic hierarchy and $A_{nv}$ is the area corresponding to the dashpot, then:

$$A_n\cdot\sigma_n(t)=N_n\cdot A_{n+1}\cdot\sigma_{n+1}(t)+A_{nv}\cdot\sigma_{nv}(t)\qquad\text{so}\qquad\sigma_n(t)=N_n\cdot\frac{A_{n+1}}{A_n}\cdot\sigma_{n+1}(t)+\frac{A_{nv}}{A_n}\cdot\sigma_{nv}(t).$$

As $A_n=N_n\cdot A_{n+1}+A_{nv}$, so we have $N_n\cdot\frac{A_{n+1}}{A_n}+\frac{A_{nv}}{A_n}=1$ The average fraction of the area occupied by a certain component in a set of parallel cross sections of a composite body can be used as an estimation of the volume fraction of the component in the body [9]. As consequence of this connection between area averages and volumes of a given phase in a body of composite material, if $\frac{A_{n+1}}{A_n}$ is interpreted as an estimation of the average fraction of parallel cross sectional areas of viscoelastic boxes of level n+1 inside a box of level, we have $\varphi_n\cong\frac{A_{n+1}}{A_n}$ An analogous derivation gives $\varphi_{nv}\cong\frac{A_{nv}}{A_n}$. Of course both stresses and strains must be interpreted also as suitable averages.

## A Transmission Line Equation for the Viscoelastic Hierarchy

From (5) we have:

$$\varphi_{nv}\cdot\sigma_{nv}(t)=\sigma_n(t)-N_n\cdot\varphi_n\cdot\sigma_{n+1}(t). \tag{7}$$

From (23), taking into account equations (2) and (4), $\varepsilon_n(t)=\varepsilon'_n(t)+\varepsilon_{n+1}(t)$ and $\sigma_{nv}(t)=\eta_n\cdot\frac{\partial}{\partial t}\varepsilon_n(t)$, as well as the constitutive relation between stress and strain for the in series spring inside boxes of level n, that is $\varepsilon'_n(t)=J_n\cdot\sigma_{n+1}(t)$ , we obtain the following system of linear differential equations:

$$\varphi_{nv}\cdot\eta_n\cdot\frac{\partial}{\partial t}\varepsilon_n(t)=\frac{(\varepsilon_{n-1}(t)-\varepsilon_n(t))}{J_{n-1}}-N_n\cdot\varphi_n\cdot\frac{(\varepsilon_n(t)-\varepsilon_{n+1}(t))}{J_n}. \tag{8}$$

The idea now is to approximate (8) by a partial differential equation of the transmission line type used in electric engineering, following a general method and employing the calculations developed in reference [10].

We take the Laplace transform in both members of (8):

$$\varphi_{nv}\cdot\eta_n\cdot s\cdot\bar{\varepsilon}_n(s)=\frac{(\bar{\varepsilon}_{n-1}(s)-\bar{\varepsilon}_n(s))}{J_{n-1}}-N_n\cdot\varphi_n\cdot\frac{(\bar{\varepsilon}_n(s)-\bar{\varepsilon}_{n+1}(s))}{J_n}. \tag{9}$$

After some reordering, (9) may be put as follows, being $T_n = \varphi_{nv} \cdot \eta_n \cdot J_{n-1}$ :

$$T_n \cdot s \cdot \bar{\varepsilon}_n(s) = \bar{\varepsilon}_{n-1}(s) - 2 \cdot \bar{\varepsilon}_n(s) + \bar{\varepsilon}_{n+1}(s) + \left(1 - \frac{N_n \cdot \varphi_n \cdot J_{n-1}}{J_n}\right) \cdot \left(\bar{\varepsilon}_n(s) - \bar{\varepsilon}_{n+1}(s)\right)$$

If the branching numbers, volume fractions and elastic compliances are constants then:

$$T_n \cdot s \cdot \bar{\varepsilon}_n(s) = \bar{\varepsilon}_{n-1}(s) - 2 \cdot \bar{\varepsilon}_n(s) + \bar{\varepsilon}_{n+1}(s) + (1 - N \cdot \varphi) \cdot \left(\bar{\varepsilon}_n(s) - \bar{\varepsilon}_{n+1}(s)\right) \ . \tag{10}$$

Following the approach of [10], let us define $\xi = \dfrac{1}{N \cdot \varphi}$ and introduce a new variable:

$$x = \left(\frac{1}{N \cdot \varphi}\right)^n = \xi^n \ . \tag{11}$$

If the parameters of the dashpots grow like $\eta_n = \left(\dfrac{1}{\varphi^\beta}\right)^n \cdot \eta$, taking into account that, from (11), $n = \dfrac{\ln x}{\ln \xi}$ we obtain:

$$\left(\frac{1}{\varphi^\beta}\right)^n = e^{n \cdot \ln\left(\frac{1}{\varphi^\beta}\right)} = e^{\frac{\ln\left(\frac{1}{\varphi^\beta}\right)}{\ln \xi} \cdot \ln x} = x^{1+v} \ . \tag{12}$$

Here, by definition:

$$v = \frac{\ln\left(N \cdot \varphi^{1-\beta}\right)}{\ln\left(\dfrac{1}{N \cdot \varphi}\right)} \ . \tag{13}$$

As we shall see in the discussion, $v$ is always positive. If we take $n$ as a continuous real variable, then $x = \xi^n$ is a regular real function of $n$ . We define: $\bar{\varepsilon}_n(s) = \tilde{\varepsilon}(x,s) = \tilde{\varepsilon}\left(\xi^n, s\right)$
Then $\bar{\varepsilon}_{n-1}(s) - \bar{\varepsilon}_n(s) = \tilde{\varepsilon}\left(\dfrac{x}{\xi}, s\right) - \tilde{\varepsilon}(x,s)$ may be approximated by $-\dfrac{x}{\xi} \cdot (\xi - 1) \cdot \dfrac{\partial}{\partial x}\tilde{\varepsilon}\left(\dfrac{x}{\xi}, s\right)$ , $\bar{\varepsilon}_n(s) - \bar{\varepsilon}_{n+1}(s) = \tilde{\varepsilon}(x,s) - \tilde{\varepsilon}(\xi \cdot x, s)$ may be approximated by $- x \cdot (\xi - 1) \cdot \dfrac{\partial}{\partial x}\tilde{\varepsilon}(x,s)$ and $\bar{\varepsilon}_{n-1}(s) - 2 \cdot \bar{\varepsilon}_n(s) + \bar{\varepsilon}_{n+1}(s)$ may be approximated by $x \cdot (\xi - 1)^2 \cdot \dfrac{\partial}{\partial x}\left(x \cdot \dfrac{\partial}{\partial x}\tilde{\varepsilon}(x,s)\right)$ . Taking all this into account, after several operations we obtain the following transmission line equation in Laplace's domain:

$$\frac{\partial^2}{\partial x^2}\tilde{\varepsilon}(x,s) - \tau_\varepsilon \cdot s \cdot x^{v-1} \cdot \tilde{\varepsilon}(x,s) \approx 0 \ . \tag{14}$$

Here, with $\xi = \dfrac{1}{N \cdot \varphi}$:

$$\tau_\varepsilon = \frac{\left(\eta_1 \cdot J \cdot N^\beta\right)}{(\xi - 1)^2 \cdot \xi^{2-\beta}} \ . \tag{15}$$

Equation (14) allows an approximate description of a finite viscoelastic hierarchy, say of N levels. This differential equation is equivalent to a modified Bessel equation and from a mathematical point of view is identical with the self-similar transmission line equation derived in [10]. The solutions of equation (14) are linear combinations of modified Bessel functions, both of the first kind $x^{\frac{1}{2}} \cdot I_p\left(2 \cdot p \cdot \sqrt{a} \cdot x^{\frac{1}{2 \cdot p}}\right)$ and of the second kind $x^{\frac{1}{2}} \cdot K_p\left(2 \cdot p \cdot \sqrt{a} \cdot x^{\frac{1}{2 \cdot p}}\right)$. The parameter $p$ verifies:

$$p = \frac{1}{1+v} = \frac{\ln\left(\frac{1}{N \cdot \varphi}\right)}{\ln\left(\frac{1}{\varphi^{\beta}}\right)}. \tag{16}$$

### Power Law Strain

When $\tau_\varepsilon \cdot s$ is positive and much less than 1, the dominant term in the formula of the Laplace's transform $\overline{Q}_1(s)$ for the viscoelastic creep kernel $Q_1(t)$ can be estimated by a suitable modification of the corresponding formula for the input impedance of an infinite self-similar transmission line that appears as **case C** in the mathematical appendix to chapter 7 in reference [10]. We obtain:

$$\overline{Q}_1(s) = J \cdot p^{1-2p} \cdot \frac{\Gamma(p)}{\Gamma(1-p)} \cdot \frac{1}{(\tau_\varepsilon \cdot s)^p}. \tag{17}$$

Here, for $x > 0$, $\Gamma(x) = \int_0^\infty e^{-u} \cdot u^{x-1} \cdot du$ is the Gamma Function. Now we don't have indeterminate parameters as we had as a result of applying the method of functional equations [8].

### Stretched Exponential Stress Relaxation

If a linear viscoelastic material has both instantaneous (elastic) $\varepsilon_0(t)$ and delayed $\varepsilon_d(t)$ strain response to an applied stress $\sigma(t)$ that is zero for negative times, the total strain $\varepsilon_0(t) + \varepsilon_d(t)$ is given by:

$$\varepsilon(t) = J_0 \cdot \sigma(t) + \int_0^t K(t-t') \cdot \sigma(t') \cdot dt'. \tag{18}$$

If the case of the composite material studied in this work, the creep kernel $K(t)$ may be identified with $Q_1(t)$, the kernel of the box in level one in the viscoelastic hierarchy. Inverting (18) and introducing an elastic impact modulus $M_0$ such that $J_0 \cdot M_0 = 1$ and a stress relaxation kernel $L(t)$:

$$\sigma(t) = M_0 \cdot \varepsilon(t) - \int_0^t L(t-t') \cdot \varepsilon(t') \cdot dt'. \tag{19}$$

Taking Laplace's transforms in (18) and (19), it is possible to relate the transform $\overline{L}(s)$ of the stress relaxation kernel with the transform $\overline{K}(s)$ of the creep kernel:

$$\frac{\overline{L}(s)}{M_0} = \frac{\left(\frac{\overline{K}(s)}{J_0}\right)}{1 + \left(\frac{\overline{K}(s)}{J_0}\right)}. \tag{20}$$

Now we substitute the Laplace's transform of the power law kernel (see equation (17)) in place of $\overline{K}(s)$ in (20) and we obtain:

$$\frac{\overline{L}(s)}{M_0}=\frac{\left(\frac{\chi}{s\cdot\tau_\varepsilon}\right)^p}{1+\left(\frac{\chi}{s\cdot\tau_\varepsilon}\right)^p}\;. \tag{21}$$

$$\chi=\left(\frac{J}{J_0}\right)^{\frac{1}{p}}\cdot p^{\frac{1}{p}-2}\cdot\left(\frac{\Gamma(p)}{\Gamma(1-p)}\right)^{\frac{1}{p}}\;. \tag{22}$$

Inverting the transform (21):

$$\frac{L(t)}{M_0}=\frac{\chi}{\tau_\varepsilon}\cdot\left(\frac{\tau_\varepsilon}{\chi\cdot t}\right)^{1-p}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma((n+1)\cdot p)}\cdot\left(\left(\frac{\chi\cdot t}{\tau_\varepsilon}\right)^p\right)^n\;. \tag{23}$$

As shown in [11], the function $\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma((n+1)\cdot p)}\cdot(u)^{p\cdot n}$ can be approximated by $p\cdot(2-p)^p\cdot e^{-p\cdot(2-p)^p\cdot u^p}$. Then:

$$L(t)\approx M_0\cdot\frac{\chi}{\tau_\varepsilon}\cdot\left(\frac{\tau_\varepsilon}{\chi\cdot t}\right)^{1-p}\cdot p\cdot(2-p)^p\cdot e^{-p\cdot(2-p)^p\cdot\left(\frac{\chi\cdot t}{\tau_\varepsilon}\right)^p}\;. \tag{24}$$

From (24) the relaxation of stress at a constant strain $\varepsilon_0$ for a viscoelastic body is given by $\sigma(t)=M(t)\cdot\varepsilon_0$ with, by defintion: $M(t)=\left(M_0-\int_0^t L(t')\cdot dt'\right)$. Taking into account (24) it follows that the stress relaxes according to a **stretched exponential**:

$$\sigma(t)=M(t)\cdot\varepsilon_0=\sigma_0\cdot e^{-\left(\frac{t}{\tau_\sigma}\right)^p}\;. \tag{25}$$

In this equation: $\sigma_0=M_0\cdot\varepsilon_0$ A formula for the characteristic stress relaxation time $\tau_\sigma$ given as function of the characteristic time $\tau_\varepsilon$ of the hyperbolic strain response follows from formulae (22) and (24):

$$\tau_\sigma=\frac{\tau_\varepsilon}{(2-p)\cdot\sqrt[p]{p}\cdot p^{\frac{1}{p}-2}\cdot\left(\frac{\Gamma(p)}{\Gamma(1-p)}\right)^{\frac{1}{p}}\cdot\left(\frac{J}{J_0}\right)^{\frac{1}{p}}}\;. \tag{26}$$

### The parameter $p$ of the power law strain response.

From (16) we conclude that $p$ must be positive, because both $\xi=\frac{1}{N\cdot\varphi}$ and $\frac{1}{\varphi^\beta}$ are greater than 1. But from both, experimental results and thermodynamic requirements related with elastic energy dissipation, we know that $p$ must be less than 1. This is equivalent to the restriction $\nu>1$. Taking into account the definition of $v$ given in equation (13), as well as the inequality $N\cdot\varphi<1$, we see that $v>1$ is verified if and only if $N\cdot\varphi^{1-\beta}>1$. The physical origin of this last inequality and the physical origin of the scale law for the dashpot viscous modulus $\eta_n\cdot\left(\varphi^{\beta}\right)^{n-1}\cong\eta_1$, can be found examining the geometric scaling laws of the viscoelastic hierarchy. Let us consider an element of volume $V_n$ in the composite material corresponding to a combination in series of an elastic element and a whole viscoelastic box of level n. Let us introduce a characteristic length $l_n$ corresponding to volume $V_n$ and a characteristic area $S_n\approx l_n^2$ of the interfaces between the volume element and its viscous matrix environment. We assume that $l_{n+1}\approx\varphi^{1/3}\cdot l_n$, so $S_{n+1}\approx\varphi^{2/3}\cdot S_n$ and $V_{n+1}\approx\varphi\cdot V_n$. Furthermore, we suppose that the force $F_n$ taken by the viscous matrix in the volume element is given by $F_n=\mu\cdot S_n\cdot\frac{\partial\varepsilon_n}{\partial t}$. Then, the force per unit volume $\hat{\sigma}_{nv}$ is given by:

$\hat{\sigma}_{nv}=\frac{F_n}{V_n}=\mu\cdot\frac{S_n}{V_n}\cdot\frac{\partial\varepsilon_n}{\partial t}=\left(\varphi^{-\frac{1}{3}}\right)^{n-1}\cdot\mu\cdot\frac{S_1}{V_1}\cdot\frac{\partial\varepsilon_n}{\partial t}$. We suppose now that the viscous stress $\sigma_{nv}$ is related with the density of volumetric viscous forces by a constant factor: $\sigma_{nv}=\gamma\cdot\hat{\sigma}_{nv}$ So that $\sigma_{nv}=\eta_n\cdot\frac{\partial\varepsilon_n}{\partial t}$, with $\eta_n=\left(\varphi^{-\frac{1}{3}}\right)^{n-1}\cdot\eta_1$ and $\eta_1=\gamma\cdot\mu\cdot\frac{S_1}{V_1}$. As consequence, we obtain the scale law of viscous dissipation $\eta_n\cdot\left(\varphi^{\beta}\right)^{n-1}\cong\eta_1$ with $\beta=\frac{1}{3}$. According to this model, the parameter of the power law and of the stretched exponential verifies $p=\frac{\ln\left(\frac{1}{N\cdot\varphi}\right)}{\ln\left(1/\varphi^{\frac{1}{3}}\right)}$. Now $v$ is positive and $p$ is less than one if $N\cdot S_{n+1}/S_n\approx N\cdot\varphi^{2/3}=\left(N\cdot\varphi^{1-\beta}\right)$ is greater than 1: this happens if the total area related with viscous dissipation increases from one level of the hierarchy to the next one.

## Discussion and Conclusions

A unified phenomenological approach to power law strain and stretched exponential stress relaxation was developed in this paper and in reference [8], for a simplified example of viscoelastic hierarchy.

An analytical formula for the power law parameter $p$ as function of the branching number and volume fractions of the hierarchical composite material was derived for a simplified example. In this case the parameter of the stretched exponential turned to be the same as the power law parameter.

The characteristic times $\tau_\varepsilon$ and $\tau_\sigma$ were given as well defined functions of the mechanical parameters of the composite material. Due to its scale invariance, the hyperbolic strain response has an infinite and unbounded set of local time scales $\frac{t}{p}$. The characteristic time $\tau_\varepsilon$ must not be considered as a time scale of the hyperbolic response. It gives a measure of the numerical order of the interval of time (since the application of a step in stress) after which the power law can be applied. The characteristic time $\tau_\sigma$ gives a measure of the numerical order of the interval of time (since the application of a step in strain) after which the stretched exponential law for stress relaxation can be applied. The stretched exponential also has an infinite and unbounded set of local time scales $\left(\tau_\sigma/t\right)^{p}\cdot t$ Contrary to the hyperbolic response case, now $\tau_\sigma$ appears in formula for the local time scale of stress relaxation [8].

If properly validated in practice, the kind of approach developed here could be used as a guide to the design of power law viscoelastic materials with desired values of the power law parameter $p$ and characteristic times $\tau_\varepsilon$ and $\tau_\sigma$.

The transmission line approach enables us to model different in series and in parallel viscoelastic hierarchies, considered as generalized quadrupoles. In this case strain corresponds to voltage and mechanical force (stress multiplied by area) corresponds to current. Dashpots correspond to capacitors and elastic elements correspond to resistances. With these correspondences, the well-developed mathematical tools of distributed parameter electric networks can be applied to study viscoelastic hierarchies. Random branching number and moduli could be introduced.

Finite viscoelastic hierarchies show, besides the high frequency cut off that corresponds to the beginning of the deformation process, a low frequency cut off for long times of deformation. In the model based in a set of nested boxes, the low frequency cut off behaviour depends of the termination (elastic elements or dashpots) of the last box in the hierarchy.

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